4.1 Maximum and Minimum Values

Learning Objectives: After completing this section, we should be able to

- find absolute extrema and local extrema of a function via its derivative.

Here are some questions we are trying to answer:

- How many items should a manufacturer make to
- What trajectory of an object
- What position gives

Here are some informal definitions. See the textbook for the formal definition if you are interested.

The point $(c, f(c))$ is a:

- **local maximum** if
- **local minimum** if
- **absolute max** or **global max** if
- **absolute min** or **global min** if

There are some fringe cases

Theorem (Extreme Value Theorem)**.** *If $f(x)$ is continuous on a closed interval $[a, b]$, then*

Note: Local minimums and maximums must be

Fact: A local extrema can only occur at x

Definition. If c is an interior point in our domain; i.e., $a < c < b$, then c is a

Question. *True or false: If $f'(c) = 0$ or does not exist, then there must be a local max or min at $x = c$.*

Question. *True or false: If a local max or min occurs at $x = c$, then $f'(c) = 0$ or does not exist.*

Question. *True or false: Maximums or minimums only occur at critical values.*

Fact: If $f(x)$ is continuous on $[a, b]$, then

Example. Let $f(x) = e^x \sin(x)$ on $[-2, 7]$. Find the absolute maximum and minimum.

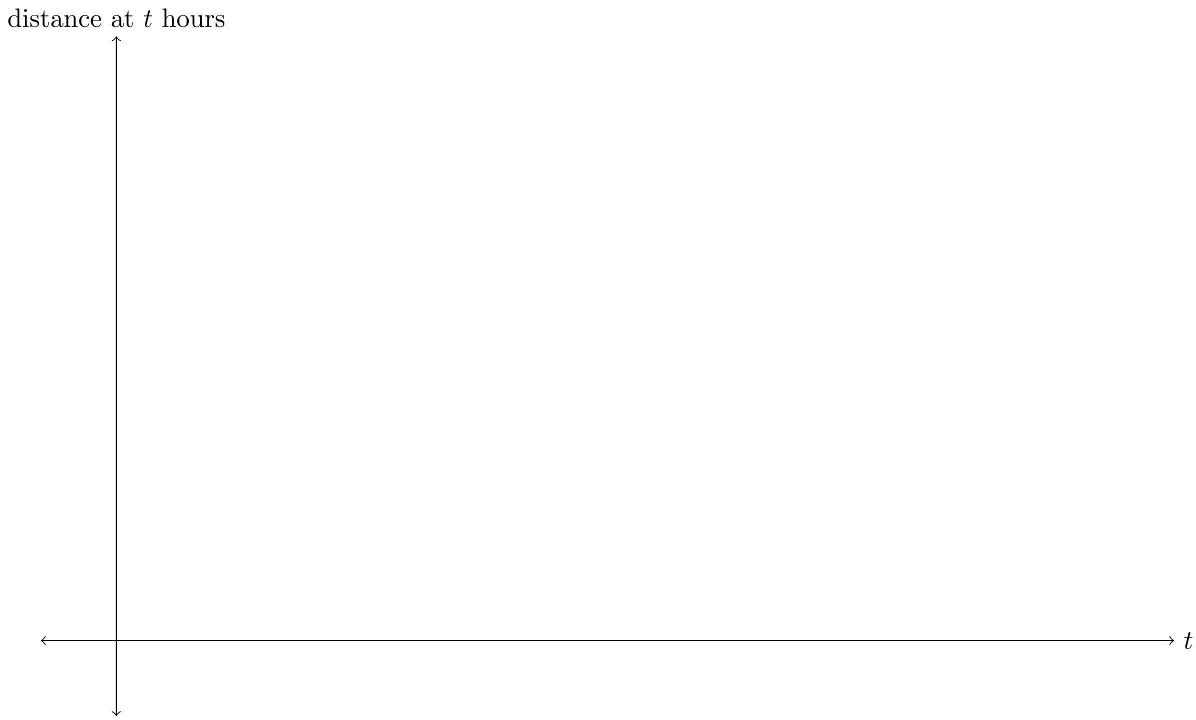
We need to be careful with our terminology.

4.2 The Mean Value Theorem

Learning Objectives: After completing this section, we should be able to

- use the Mean Value Theorem and apply it to prove other results.

Example. Suppose you are driving on a highway. You note that you have travelled 100 miles in the last 2 hours. What do you know about your instantaneous velocity at any point on the trip?



At some point,

Theorem (Mean Value Theorem)**.** *If f is continuous over $[a, b]$ and*

Example. Let $f(x) = |x|$. Note $f(-1) = 1$ and $f(2) = 2$.

Example. Let $f(x) = \sqrt{x}$. Note $f(0) = 0$ and $f(4) = 2$.

Example. Suppose f is continuous on $[a, b]$ and

Example. In Iowa, there are marks on the interstate highway every 0.1 miles visible to an airplane or helicopter. The speed limit on the interstate is 70 mph. A police helicopter notices that a car crosses one mark and then 4.8 seconds later the car crosses the next mark. Will the driver get a speeding ticket?

4.3 Derivatives and Shapes of Graphs

Learning Objectives: After completing this section, we should be able to

- find the intervals of increase and decrease of a function using the first derivative of the function.
- find the intervals of concavity of a function using the second derivative of the function.

4.3.1 First Derivative

Definition. *Increasing/Decreasing*

- If $f'(x) > 0$ on
- If $f'(x) < 0$ on

Example.

Fact: The only places f changes from increasing to decreasing

The first derivative test for local extrema: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) except

- If $f'(x)$ changes
- If $f'(x)$ changes
- If $f'(x)$ doesn't change

Example. Let $f(x) = 2x^3 + 3x^2 - 12x + 1$. Find all local extrema.

You try!

Example. Let $f(x) = 3x^5 - 20x^3$. Find all local extrema.

Example. Let $f(x) = x^2 - 2\ln(x)$. Find all local extrema.

4.3.2 Second Derivative

Consider the second derivative:

Definition. If $f'(x)$ is increasing, then f is

Definition. If $f'(x)$ is decreasing, then f is

Definition. If f changes concavity at $x = c$, then

Example. Determine the intervals of concavity for $f(x) = x^4 - 2x^3 + 1$.

Example Continued.

You try!

Example. Find the intervals of concavity and the inflection points for $f(x) = 2x^4 + 8x^3 + 12x^2 - x - 2$.

Second derivative test for local extrema: What can you say about extrema when $f(x)$ is concave up? Down?

Second derivative test: Suppose $(c, f(c))$ is a critical point. Then,

- If $f''(c) > 0$, then
- If $f''(c) < 0$, then
- If $f''(c) = 0$, then

Example. From before: Let $f(x) = 2x^3 + 3x^2 - 12x + 1$. Use the second derivative test to find extrema.

You try!

Example. Let $f(x) = 2x^2 \ln(x) - 11x^2$. Find all local extrema using the second derivative test.

Question. *Should you use the first derivative test or the second derivative test?*

4.4 Indeterminate Forms and L'Hopital's Rule

Learning Objectives: After completing this section, we should be able to

- apply L'Hopital's Rule to evaluate the limit of an expression in an indeterminate form.

Recall that

Remember, when talking about indeterminate forms,

Theorem (L'Hopital's Rule). *If* $\lim_{x \to c} \dfrac{f(x)}{g(x)}$ *is*

Important!

- This is not

- Only applies

Example. $\lim_{x \to \infty} \dfrac{x+1}{x^2-5}$

Example. $\lim_{x \to 0} \frac{\sin(x)}{x}$

You try!

Example. $\lim_{x \to 0} \frac{\sin(7x)}{4x}$

You try!

Example. $\lim_{x \to \infty} \frac{3x^2 + 2x - 1}{8x^2 + 100}$

Example. $\lim_{x\to 0} \frac{x-2}{x^2+4}$

Example. $\lim_{x\to 0} \frac{x\sin(x)}{1-\cos(x)}$

Example. $\lim_{x\to \infty} \frac{\ln(x)}{2\sqrt{x}}$

You try!

Example. $\lim_{x\to\infty} \frac{x}{(\ln(x))^2}$

What about other indeterminate forms?

Consider the indeterminate form $0 \cdot \infty$:

- Need to

- Note:

Example. $\lim_{x\to 0^+} x\ln(x)$

You try!

Example. $\lim\limits_{x\to\infty} e^{-x}x^2$

Example. $\lim\limits_{x\to\frac{\pi}{2}^-} \left(\frac{\pi}{2} - x\right)\tan(x)$

4.7 Optimization Problems

Learning Objectives: After completing this section, we should be able to

- convert an optimization problem in words into a mathematical optimization problem.
- solve an optimization problem.

Now that we have tools to find extrema, we can use them to solve real-world problems!

Example. Suppose x and y are 2 numbers. Find these two positive numbers satisfying the equation $xy = 3$ and the sum $x + 2y$ is as small as possible.

Example. A rectangular pen is being built against the side of a barn. There is 1000 m of fencing available. What dimensions of the pen maximize the area of the pen?

Example. A rancher is building 2 adjacent, rectangular pens against a barn, each with an area of 50 m^2. What are the dimensions of each pen that minimize the amount of fence that must be used?

4.9 Antiderivatives

Learning Objectives: After completing this section, we should be able to

- find antiderivatives of given functions.

We've spent the majority of the semester taking derivatives. How do we undo taking a derivative?

Definition. An **antiderivative** of $f(x)$ is

They are not unique!

Example. $\frac{d}{dx}(2x^3 + 4) = 6x^2$.

Antiderivatives come in in a 1-parameter family.

Notation:

Example. $\int 6x^2 dx = 2x^3 + C$

Let's find antiderivatives for basic functions.

1. **Powers**

   Example. $\int x^5 dx$

2. **Constant Multiple Rule**

   Example. $\int 6x^2 dx$

3. **Sum Rule**

   Example. $\int (6x^2 + x^5)dx$

4. **Trig Rules**
   - $\int \cos(x)dx$
   - $\int \sin(x)dx$
   - $\int \sec^2(x)dx$
   - $\int \csc^2(x)dx$
   - $\int \sec(x)\tan(x)dx$
   - $\int \csc(x)\cot(x)dx$

5. **Inverse Trig**
   - $\int \frac{1}{1+x^2}dx$
   - $\int \frac{1}{\sqrt{1-x^2}}dx$
   - $\int -\frac{1}{\sqrt{1-x^2}}dx$

6. **Logs and Exponentials**

7. **Constant Chains**

Example. $\int \left(x + 14 - \sqrt{x^3} + 3x^{-6} - \frac{2}{x} + \cos(4x) - \sec^2(6x)/8 + \frac{1}{3}e^{-x} - \frac{4}{1+x^2} + \pi^x \right) dx$